The nonlinear Schrödinger equation on metric graphs

Oberseminar Geometrische Analysis

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Christophe Troestler (UMONS, Mons, Belgium),
Simone Dovetta and Enrico Serra (Politecnico di Torino, Italy)

Tuesday 8 July 2025

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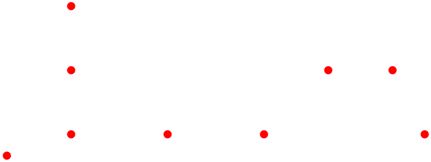
- you!

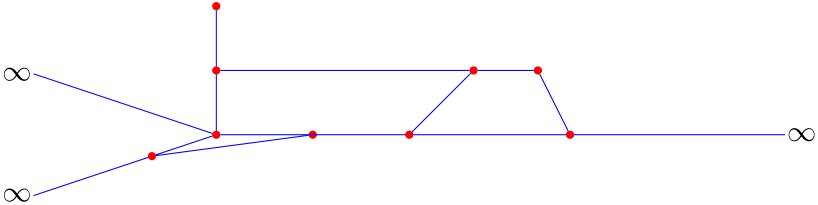
What is a metric graph?

A metric graph is made of vertices



What is a metric graph?

A metric graph is made of vertices and of edges joining the vertices or going to infinity.



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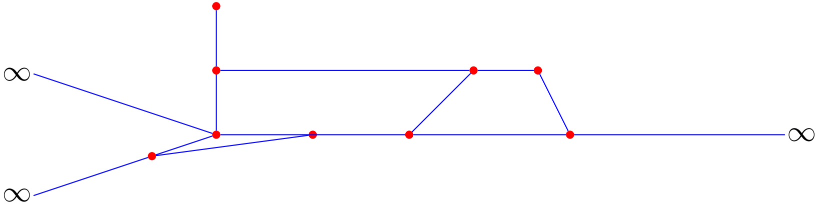

- *metric* graphs: the lengths of edges are important.

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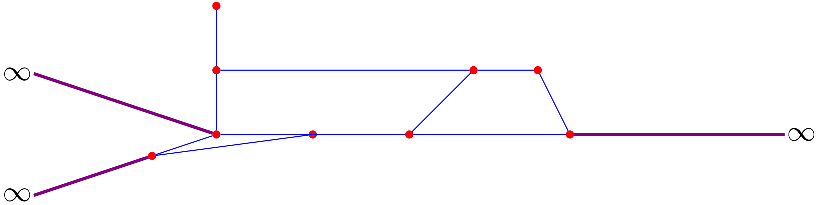

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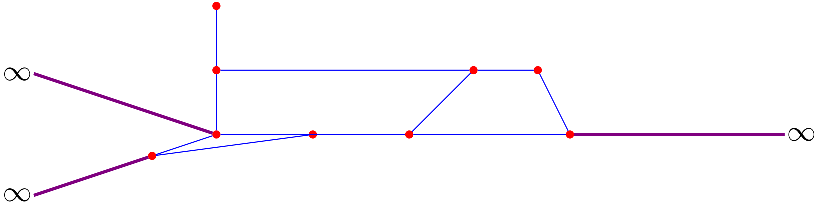
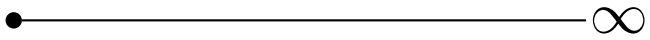


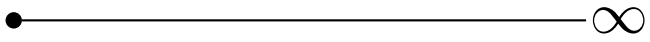
- *metric* graphs: the lengths of edges are important.
- the edges going to infinity are halflines and have *infinite length*.
- a metric graph is *compact* if and only if it has a finite number of edges of finite length.

Constructions based on halflines



The halfline

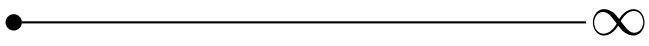
Constructions based on halflines



The halfline

The line

Constructions based on halflines



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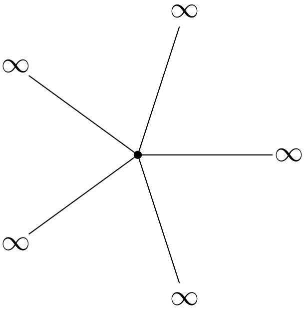
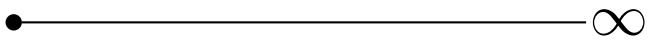


The 5-star graph

Constructions based on halflines



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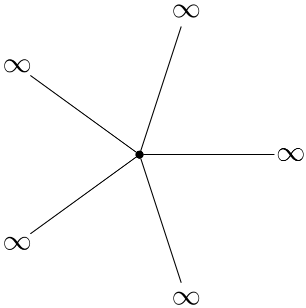

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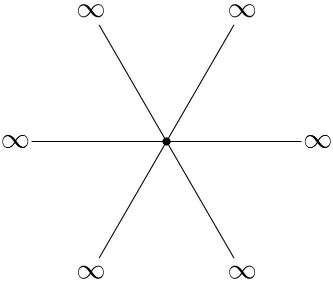
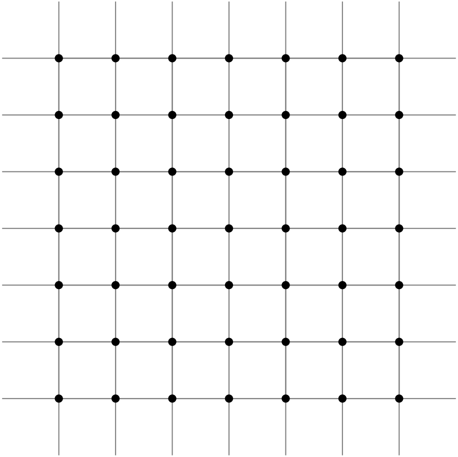
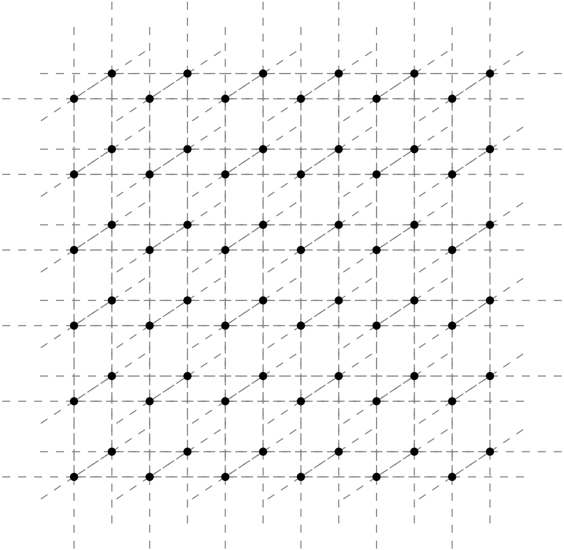


The 6-star graph

Periodic graphs



The two-dimensional grid



The three-dimensional grid

Infinite trees

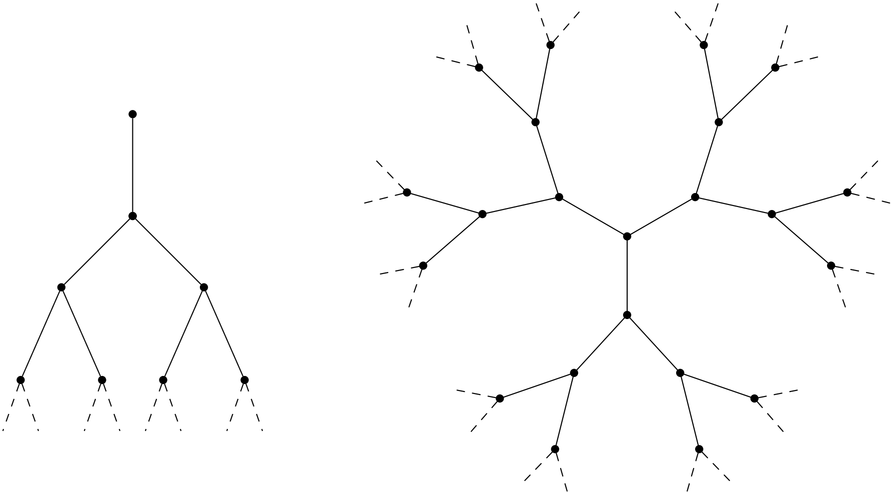
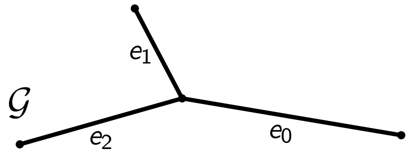


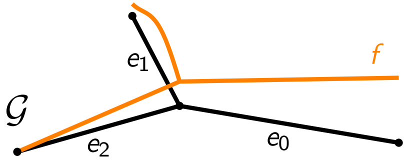
Figure: Infinite trees

Functions defined on metric graphs



A metric graph $\mathcal{G}$ with three edges e_0 (length 5), e_1 (length 4) and e_2 (length 3)

Functions defined on metric graphs



A metric graph $\mathcal{G}$ with three edges e_0 (length 5), e_1 (length 4) and e_2 (length 3),
a function $f : \mathcal{G} \to \mathbb{R}$

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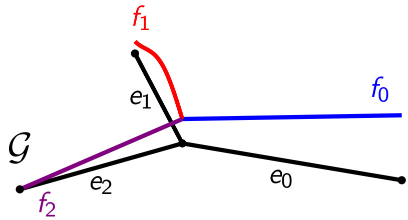

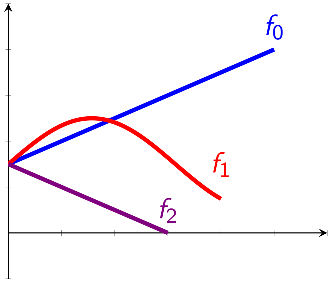

A metric graph $\mathcal{G}$ with three edges e_0 (length 5), e_1 (length 4) and e_2 (length 3), a function $f : \mathcal{G} \to \mathbb{R}$, and the three associated real functions.

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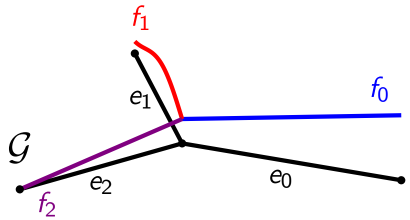

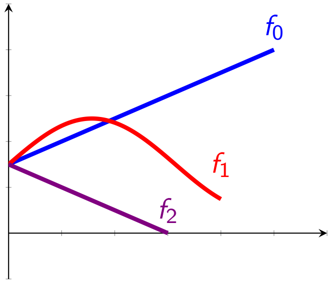

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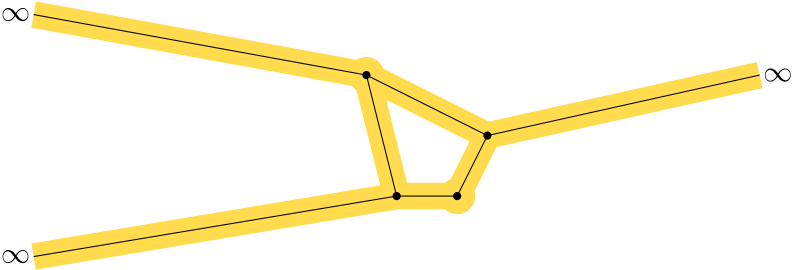
$$\int_{\mathcal{G}} f \, \mathrm{d}x := \int_0^5 f_0(x) \, \mathrm{d}x + \int_0^4 f_1(x) \, \mathrm{d}x + \int_0^3 f_2(x) \, \mathrm{d}x$$

Why studying metric graphs?

Physical motivations

Modeling structures where *only one spatial direction is important.*



A “fat graph” and the underlying metric graph

The differential system

Given constants $p > 2$ and $\lambda > 0$, we are interested in solutions $u \in L^2(\mathcal{G})$ of the differential system

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$$\begin{cases} -u'' + \lambda u = |u|^{p-2}u \quad \text{on each edge } e \text{ of } \mathcal{G}, \\ \\ \\ \end{cases} \qquad (\mathrm{NLS}_{\mathcal{G}})$$

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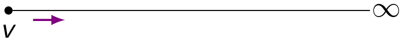
The condition on the sum of derivatives is called *Kirchhoff's condition*.

The nonlinear Schrödinger equation

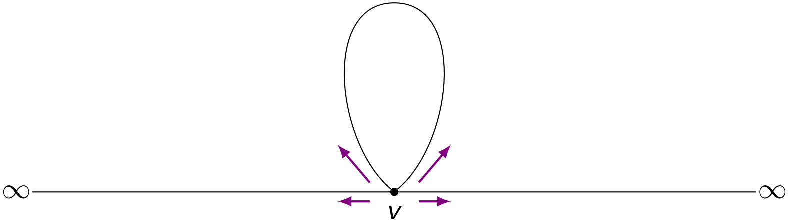
Kirchhoff's condition: degree-one nodes



$$\lim_{\substack{t \to 0 \\ t>0}} \frac{u(v+t)-u(v)}{t} = 0$$

The nonlinear Schrödinger equation

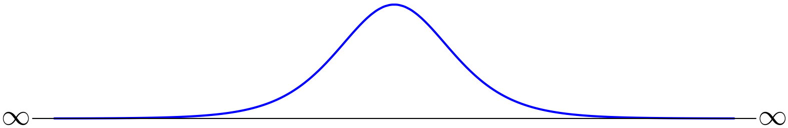
Kirchhoff's condition in general: outgoing derivatives



$$\sum_{e \succ v} \frac{\mathrm{d}u}{\mathrm{d}x_e}(v) = 0$$

The nonlinear Schrödinger equation

The soliton φ_λ on the real line



The set of nonzero solutions of $(\mathrm{NLS}_\mathcal{G})$ is

$$\mathcal{S}_\lambda(\mathbb{R}) = \left\{ \pm\varphi_\lambda(x+a) \;\middle|\; a \in \mathbb{R} \right\}$$

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$$-u'' + \lambda u = |u|^{p-2}u.$$

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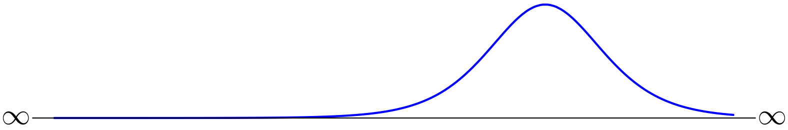

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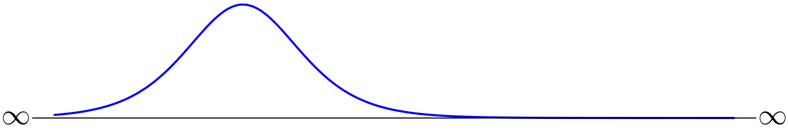

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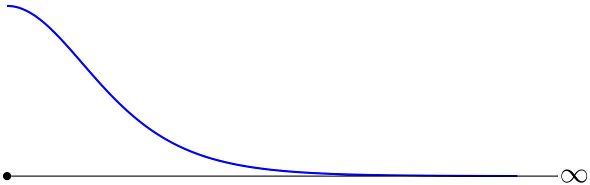
The nonlinear Schrödinger equation

The half-line: $\mathcal{G} = \mathbb{R}^+$

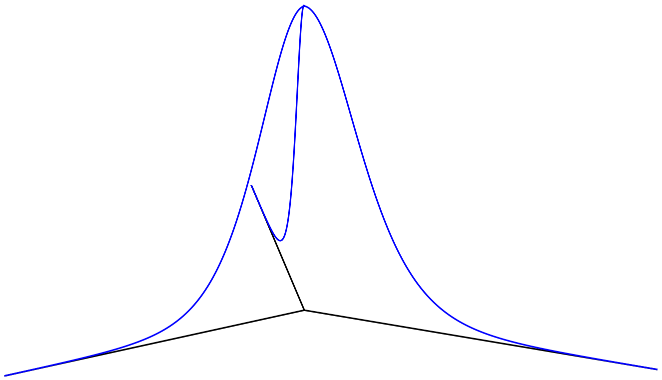


$$\mathcal{S}_\lambda(\mathbb{R}^+) = \left\{ \pm\varphi_\lambda(x)_{|\mathbb{R}^+} \right\}$$

Solutions are *half-solitons*: no more translations!

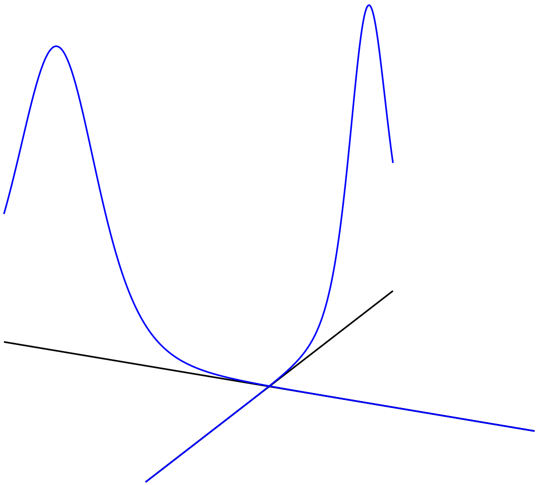
The nonlinear Schrödinger equation

The positive solution on the 3-star graph



The nonlinear Schrödinger equation

A continuous family of solutions on the 4-star graph



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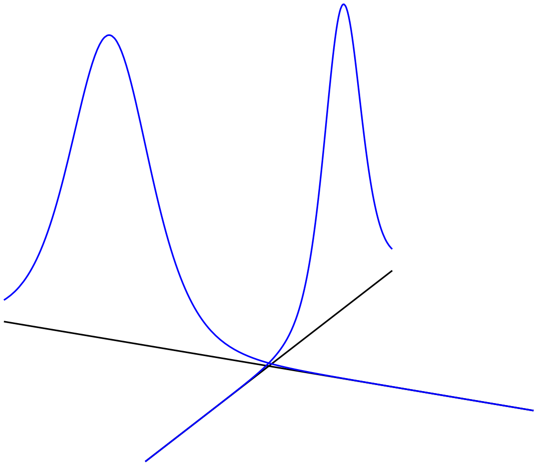

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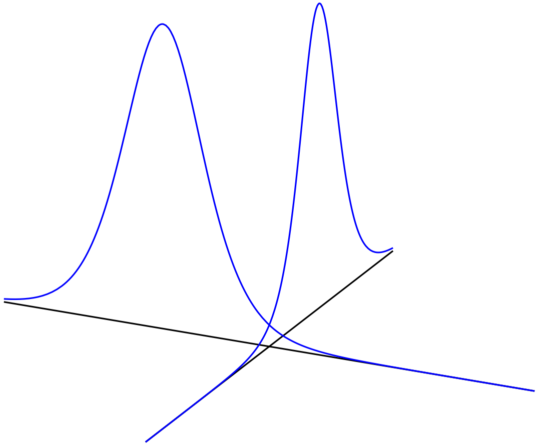

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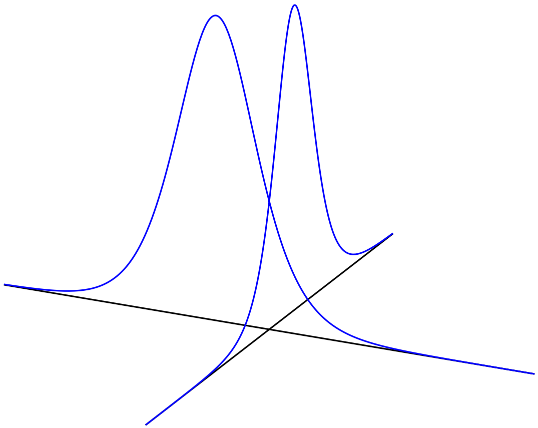

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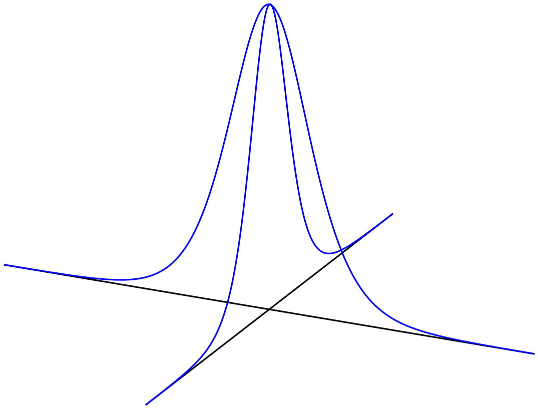

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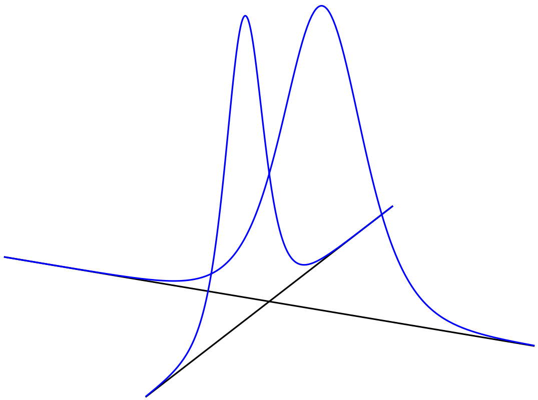

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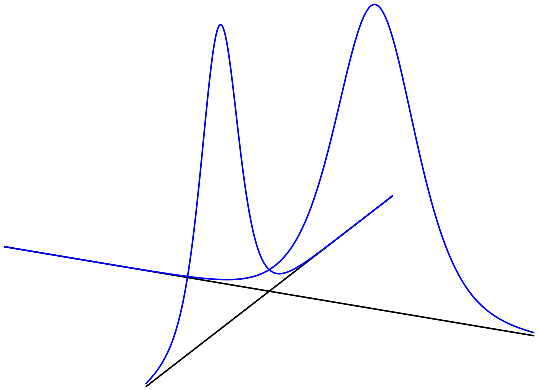

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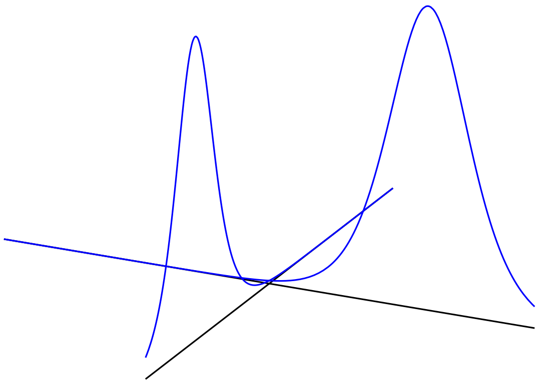

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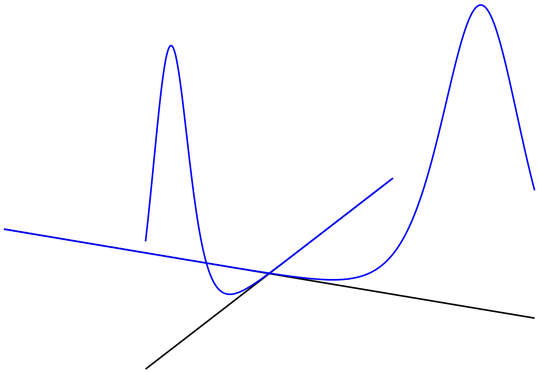

Variational formulation

We work on the Sobolev space

$$H^1(\mathcal{G}) := \Big\{ u : \mathcal{G} \to \mathbb{R} \mid u \text{ is continuous}, u, u' \in L^2(\mathcal{G}) \Big\}.$$

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Solutions of $(\mathrm{NLS}_{\mathcal{G}})$ correspond to critical points of the *action functional*

$$J_\lambda(u) := \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} + \frac{\lambda}{2}\|u\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{G})}.$$

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The level of the soliton φ_λ plays an important role in our analysis:

$$s_\lambda := J_\lambda(\varphi_\lambda).$$

The Euler-Lagrange equation associated to J_λ

The differential of $J_\lambda : H^1(\mathcal{G}) \to \mathbb{R}$ is given by

$$J_\lambda'(u)[v] = \int_{\mathcal{G}} u'(x)v'(x)\,\mathrm{d}x + \lambda \int_{\mathcal{G}} u(x)v(x)\,\mathrm{d}x - \int_{\mathcal{G}} |u(x)|^{p-2}u(x)v(x)\,\mathrm{d}x$$

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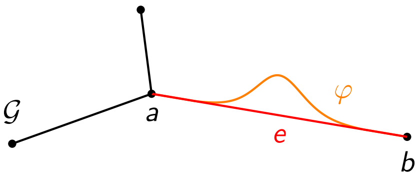
If φ has compact support in the interior of an edge $e = \mathrm{AB}$, we have...



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$$\begin{aligned} 0 &= J_\lambda'(u)[\varphi] \\ &= \int_e u'(x)\varphi'(x)\,\mathrm{d}x + \lambda \int_e u(x)\varphi(x)\,\mathrm{d}x - \int_e |u(x)|^{p-2}u(x)\varphi(x)\,\mathrm{d}x \end{aligned}$$

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$$\begin{aligned} 0 &= J_\lambda'(u)[\varphi] \\ &= \int_e u'(x)\varphi'(x)\,\mathrm{d}x + \lambda \int_e u(x)\varphi(x)\,\mathrm{d}x - \int_e |u(x)|^{p-2}u(x)\varphi(x)\,\mathrm{d}x \\ &= \frac{\mathrm{d}u}{\mathrm{d}x_e}(b)\underbrace{\varphi(b)}_{=0} - \frac{\mathrm{d}u}{\mathrm{d}x_e}(a)\underbrace{\varphi(a)}_{=0} \\ &\quad + \int_e \left(-u''(x) + \lambda u(x) - |u(x)|^{p-2}u(x)\right)\varphi(x)\,\mathrm{d}x. \end{aligned}$$

The Euler-Lagrange equation associated to J_λ

The differential of $J_\lambda : H^1(\mathcal{G}) \to \mathbb{R}$ is given by

$$J_\lambda'(u)[v] = \int_{\mathcal{G}} u'(x)v'(x)\,\mathrm{d}x + \lambda \int_{\mathcal{G}} u(x)v(x)\,\mathrm{d}x - \int_{\mathcal{G}} |u(x)|^{p-2}u(x)v(x)\,\mathrm{d}x$$

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&= \frac{\mathrm{d}u}{\mathrm{d}x_e}(b)\underbrace{\varphi(b)}_{=0} - \frac{\mathrm{d}u}{\mathrm{d}x_e}(a)\underbrace{\varphi(a)}_{=0} \\
&\quad + \int_e (-u''(x) + \lambda u(x) - |u(x)|^{p-2}u(x))\varphi(x)\,\mathrm{d}x.
\end{aligned}$$

so that $-u'' + \lambda u = |u|^{p-2}u$ on edges of $\mathcal{G}$.

Kirchhoff's condition

Let A be a vertex of $\mathcal{G}$ and let $\mathrm{B}_1, \ldots, \mathrm{B}_D$ be the vertices adjacent to A.

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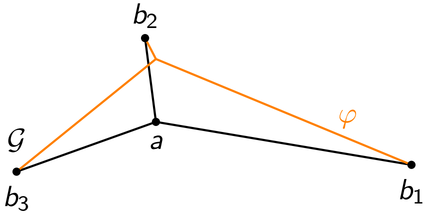

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$$\begin{aligned} 0 &= J'_\lambda(u)[\varphi] \\ &= \sum_{1 \le i \le D} \left(\int_{e_i} u' \varphi' \,\mathrm{d}x + \lambda \int_{e_i} u \varphi \,\mathrm{d}x - \int_{e_i} |u|^{p-2} u \varphi \,\mathrm{d}x \right) \end{aligned}$$

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&= \sum_{1 \le i \le D} \left(\frac{\mathrm{d}u}{\mathrm{d}x_{e_i}}(b_i) \underbrace{\varphi(b_i)}_{=0} - \frac{\mathrm{d}u}{\mathrm{d}x_{e_i}}(a_i) \underbrace{\varphi(a)}_{=1} \right)
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$$\begin{aligned}
0 &= J_\lambda'(u)[\varphi] \\
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&= \sum_{1\leq i\leq D} \left(\frac{\mathrm{d}u}{\mathrm{d}x_{e_i}}(b_i) \underbrace{\varphi(b_i)}_{=0} - \frac{\mathrm{d}u}{\mathrm{d}x_{e_i}}(a_i) \underbrace{\varphi(a)}_{=1} \right) \\
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\end{aligned}$$

so that $\sum_{1\leq i\leq D} \frac{\mathrm{d}u}{\mathrm{d}x_{e_i}}(\mathrm{A}_i) = 0$, which is Kirchhoff's condition.

The Nehari manifold

The functional J_λ is not bounded from below on $H^1(\mathcal{G})$, since if $u \neq 0$ then

$$J_\lambda(tu) = \frac{t^2}{2}\|u'\|^2_{L^2(\mathcal{G})} + \frac{\lambda t^2}{2}\|u\|^2_{L^2(\mathcal{G})} - \frac{t^p}{p}\|u\|^p_{L^p(\mathcal{G})} \xrightarrow[t\to\infty]{} -\infty.$$

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A common strategy is to introduce the *Nehari manifold* $\mathcal{N}_\lambda(\mathcal{G})$, defined by

$$\begin{aligned}\mathcal{N}_\lambda(\mathcal{G}) &:= \left\{u \in H^1(\mathcal{G}) \setminus \{0\} \mid J'_\lambda(u)[u] = 0\right\}\\ &= \left\{u \in H^1(\mathcal{G}) \setminus \{0\} \mid \|u'\|^2_{L^2(\mathcal{G})} + \lambda\|u\|^2_{L^2(\mathcal{G})} = \|u\|^p_{L^p(\mathcal{G})}\right\}.\end{aligned}$$

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If $u \in \mathcal{N}_\lambda(\mathcal{G})$, then

$$J_\lambda(u) = \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^p_{L^p(\mathcal{G})}.$$

In particular, J_λ is bounded from below on $\mathcal{N}_\lambda(\mathcal{G})$.

Action ground states

- "Ground state" action level:

$$\mathcal{J}_{\mathcal{G}}(\lambda) := \inf_{u \in \mathcal{N}_\lambda(\mathcal{G})} J_\lambda(u)$$

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- *Ground state*: function $u \in \mathcal{N}_\lambda(\mathcal{G})$ with level $\mathcal{J}_{\mathcal{G}}(\lambda)$. If it exists, it is a solution of the differential system $(\mathrm{NLS}_{\mathcal{G}})$.

A very useful tool: cutting solitons on halflines

Proposition

Assume that $\mathcal{G}$ has at least one halfline. Then,

$$\mathcal{J}_{\mathcal{G}}(\lambda) \leq s_\lambda := J_\lambda(\varphi_\lambda)$$

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Proof.

$\square$

An existence Theorem

Theorem (Adami-Serra-Tilli 2015, Dovetta-De Coster-G.-Serra-Troestler 2024)

Let $\mathcal{G}$ be a metric graph with finitely many edges, including at least one halfline. Let $p > 2$ and $\lambda > 0$ be real numbers. If

$$\mathcal{J}_{\mathcal{G}}(\lambda) < s_\lambda,$$

then action ground states exist.

Some graphs which admit action ground states

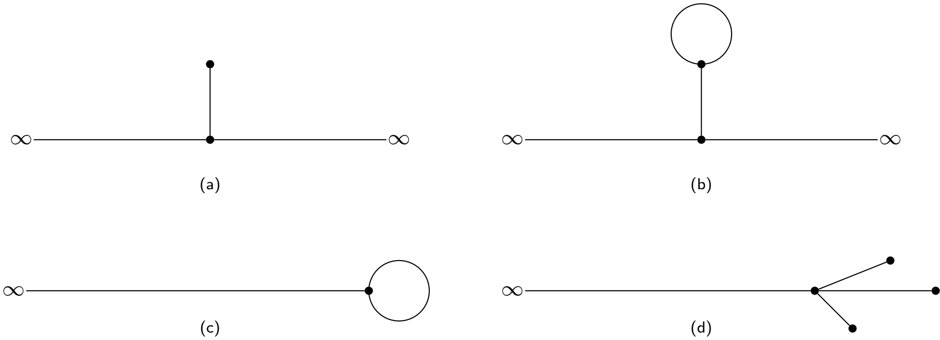
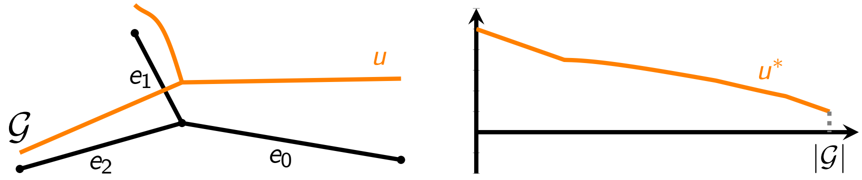


Figure: Examples of graphs admitting action ground states. (a): the $\mathcal{T}$-graph; (b): the signpost; (c): the tadpole; (d): the 3-fork.

Decreasing rearrangement on the halfline



For all $1 \leq p \leq +\infty$,

$$\|u\|_{L^p(\mathcal{G})} = \|u^*\|_{L^p(0,|\mathcal{G}|)}.$$

The Pólya–Szegő inequality

Theorem

Let $u \in H^1(\mathcal{G})$ be a nonnegative function. Then its decreasing rearrangement u^* belongs to $H^1(0, |\mathcal{G}|)$, and one has

$$\|(u^*)'\|_{L^2(0,|\mathcal{G}|)} \leq \|u'\|_{L^2(\mathcal{G})}.$$

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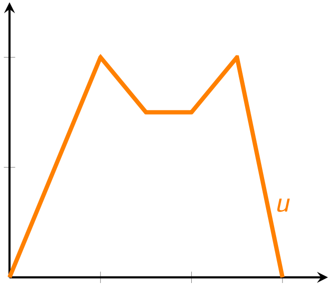
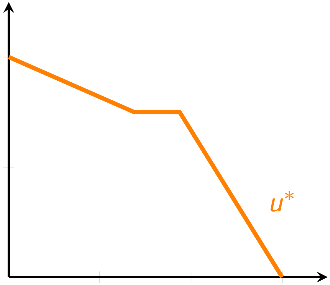
- Friedlander, L. *Extremal properties of eigenvalues for a metric graph.* Ann. Inst. Fourier (Grenoble) **55** (2005) no. 1, 199–211.

The Pólya–Szegő inequality

A simple case: affine functions

We assume that u is piecewise affine.





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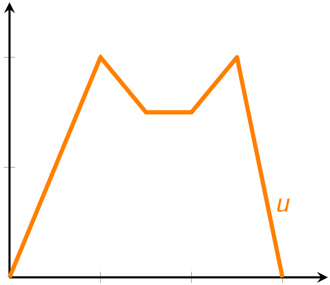

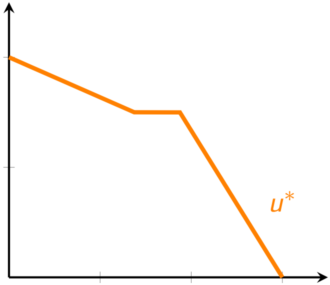

We consider a small open interval $I \subseteq u(\mathcal{G})$ so that $u^{-1}(I)$ consists of a disjoint union of open intervals on which u is affine.

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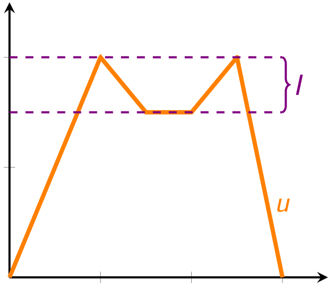

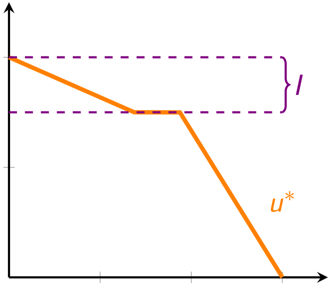

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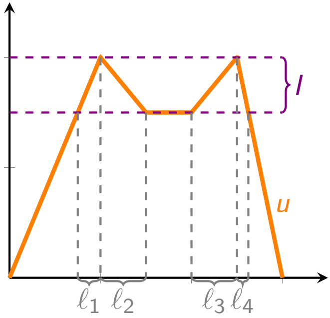

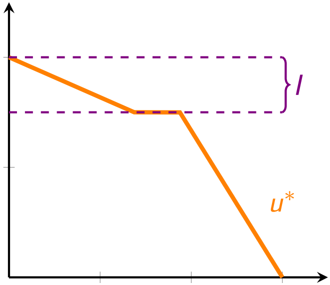

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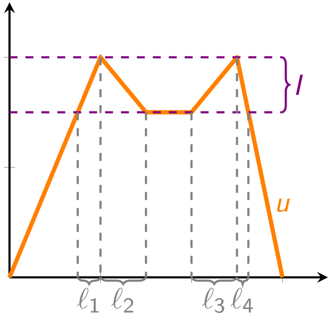

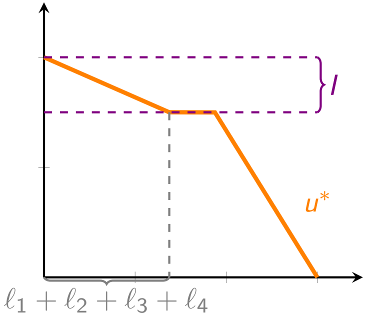

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The Pólya–Szegő inequality

A simple case: affine functions

Original contribution to $\|u'\|_{L^2}^2$:

$$A := \ell_1 \frac{|I|^2}{\ell_1^2} + \ell_2 \frac{|I|^2}{\ell_2^2} + \ell_3 \frac{|I|^2}{\ell_3^2} + \ell_4 \frac{|I|^2}{\ell_4^2}$$

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Contribution to $\|(u^*)'\|_{L^2}^2$:

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$$\frac{\ell_1 + \ell_2 + \ell_3 + \ell_4}{4} \geq \frac{4}{\frac{1}{\ell_1} + \frac{1}{\ell_2} + \frac{1}{\ell_3} + \frac{1}{\ell_4}}$$

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A consequence of the rearrangement technique

Proposition

Let $\mathcal{G}$ be a metric graph with finitely many edges, including at least one halfline. Let $p > 2$ and $\lambda > 0$ be real numbers. Then,

$$\mathcal{J}_{\mathcal{G}}(\lambda) \geq \frac{1}{2} J_\lambda(\varphi_\lambda).$$

A consequence of the rearrangement technique

Proof.

One may assume that $u \geq 0$.

A consequence of the rearrangement technique

Proof.

One may assume that $u \geq 0$. Then,

$$\begin{aligned} \|u^*\|_{L^2(0,+\infty)} &= \|u\|_{L^2(\mathcal{G})}, \\ \|u^*\|_{L^p(0,+\infty)} &= \|u\|_{L^p(\mathcal{G})}, \\ \|(u^*)'\|_{L^2(0,+\infty)} &\leq \|u'\|_{L^2(\mathcal{G})}. \end{aligned}$$

$\square$

A consequence of the rearrangement technique

Proof.

One may assume that $u \geq 0$. Then,

$$\|u^*\|_{L^2(0,+\infty)} = \|u\|_{L^2(\mathcal{G})},$$
$$\|u^*\|_{L^p(0,+\infty)} = \|u\|_{L^p(\mathcal{G})},$$
$$\|(u^*)'\|_{L^2(0,+\infty)} \leq \|u'\|_{L^2(\mathcal{G})}.$$

Then, one shows that for a suitable $t > 0$, the function tu^* belongs to $\mathcal{N}_\lambda(0,+\infty)$ and is such that

$$J_{\lambda,\mathcal{G}}(u) \geq J_{\lambda,[0,+\infty[}(tu^*).$$

$\square$

A refined Pólya–Szegő inequality...

... or the importance of the number of preimages

Theorem

Let $u \in H^1(\mathcal{G})$ *be a nonnegative function. Let* $N \geq 1$ *be an integer. Assume that, for almost every* $t \in]0, \|u\|_\infty[$*, one has*

$$u^{-1}(\{t\}) = \{x \in \mathcal{G} \mid u(x) = t\} \geq N.$$

Then one has

$$\|(u^*)'\|_{L^2(0,|\mathcal{G}|)} \leq \frac{1}{N}\|u'\|_{L^2(\mathcal{G})}.$$

Assumption (H)

Definition (Adami, Serra, Tilli (Calc. Var. PDEs. 2014))

We say that a metric graph $\mathcal{G}$ satisfies assumption (H) if, for every point $x_0 \in \mathcal{G}$, there exist two injective curves $\gamma_1, \gamma_2 : [0, +\infty[\to \mathcal{G}$ parameterized by arclength, with disjoint images except for an at most countable number of points, and such that $\gamma_1(0) = \gamma_2(0) = x_0$.

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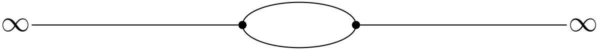

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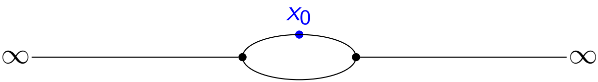

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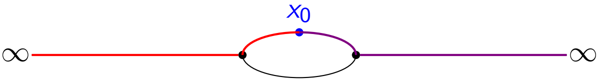

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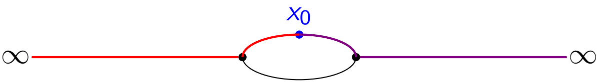

Consequence: *all* nonnegative $H^1(\mathcal{G})$ functions have at least two preimages for almost every $t \in]0, \|u\|_\infty[$.

Non-existence of ground states

Theorem (Adami, Serra, Tilli (Calc. Var. PDEs. 2014))

If a metric graph $\mathcal{G}$ satisfies assumption (H), then

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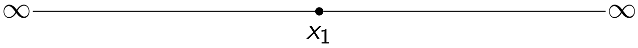
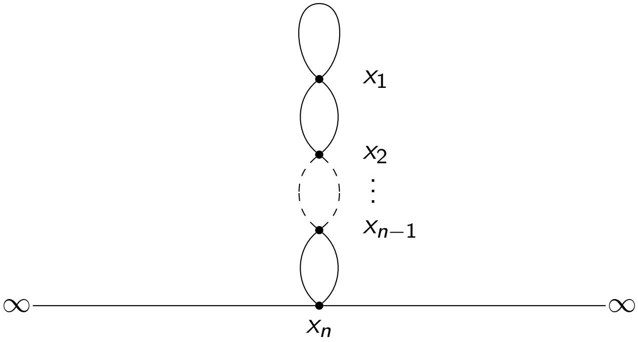
Non-existence of ground states

Exceptional graphs: the real line



Non-existence of ground states

Exceptional graphs: the real line with a tower of circles



Another action level

- Minimal level attained by the solutions of $(\mathrm{NLS}_{\mathcal{G}})$:

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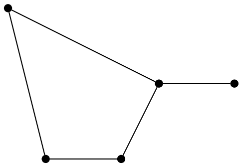
Theorem (De Coster, Dovetta, G., Serra (Calc. Var. PDEs. 2023))

For every $p > 2$, every $\lambda > 0$, and every choice of alternative between A1, A2, B1, B2, there exists a metric graph $\mathcal{G}$ where this alternative occurs.

Case A1

$\mathcal{J}_{\mathcal{G}}(\lambda) = \sigma_\lambda(\mathcal{G})$ and both infima are attained



Compact graphs

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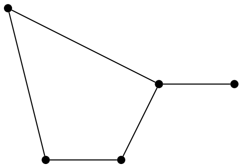

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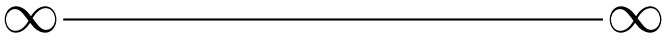

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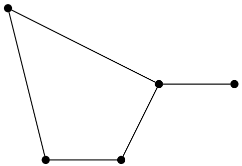

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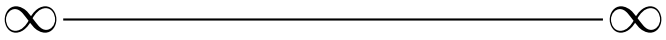

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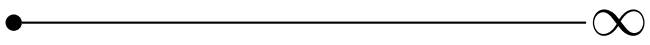

The halfline

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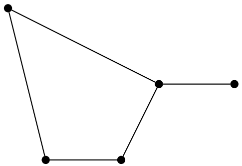

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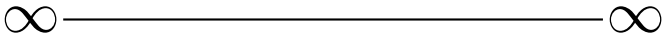

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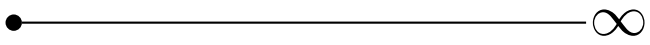

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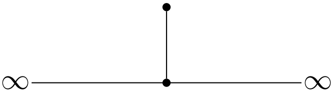

All graphs with $\mathcal{J}_{\mathcal{G}}(\lambda) < s_\lambda$

Case B1

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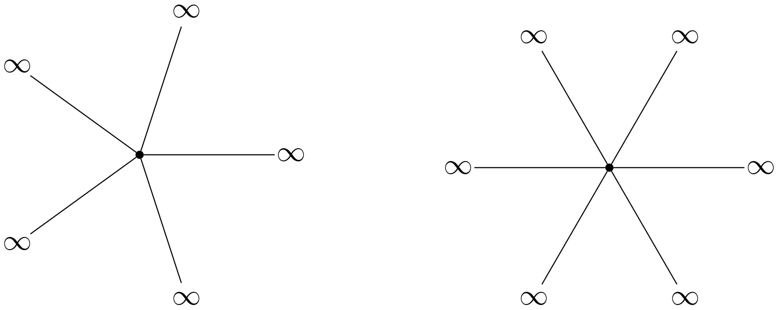
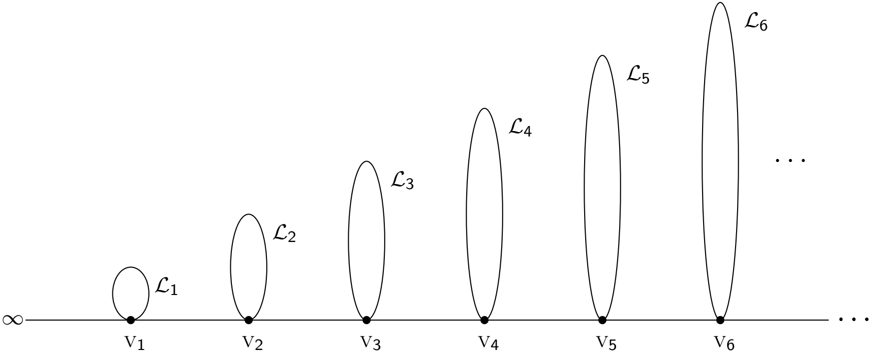


N-star graphs, $N \geq 3$

$$s_\lambda = \mathcal{J}_{\mathcal{G}}(\lambda) < \sigma_\lambda(\mathcal{G}) = \frac{N}{2} s_\lambda$$

Case A2

$\mathcal{J}_{\mathcal{G}}(\lambda) = \sigma_\lambda(\mathcal{G})$ and neither infima is attained



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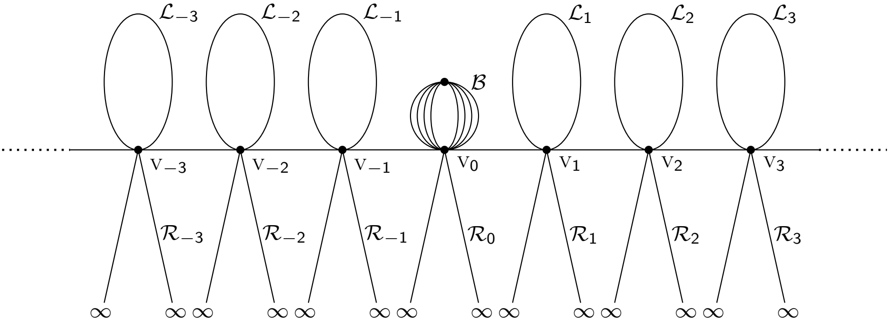

$$s_\lambda = \mathcal{J}_{\mathcal{G}}(\lambda) < \sigma_\lambda(\mathcal{G})$$

Why studying metric graphs?

Mathematical motivations

Main message

Metric graphs allow to study interesting *one dimensional* problems and are much richer than the usual class of intervals of $\mathbb{R}$.

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Replacing $\mathcal{G}$ by noncompact smooth open sets $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$ and $H^1(\mathcal{G})$ by $H^1(\Omega)$ or $H^1_0(\Omega)$, one expects that the four cases A1, A2, B1, B2 actually occur.

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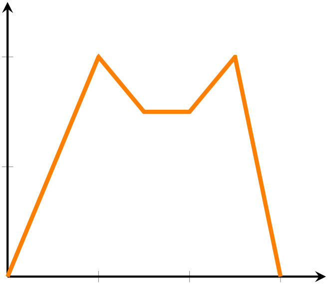
Danke schön!



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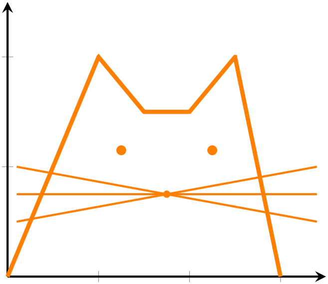

References

Adami R., Serra E., Tilli P., *NLS ground states on graphs*, Calc. Var. 54, 743–761 (2015).

Adami, R., Serra, E., Tilli, P. (2015). Lack of Ground State for NLSE on Bridge-Type Graphs. In: Mugnolo, D. (eds) Mathematical Technology of Networks. Springer Proceedings in Mathematics & Statistics, vol 128. Springer, Cham. https://doi.org/10.1007/978-3-319-16619-3_1

References

De Coster C., Dovetta S., Galant D., Serra E. *On the notion of ground state for nonlinear Schrödinger equations on metric graphs.* Calc. Var. 62, 159 (2023).

De Coster C., Dovetta S., Galant D., Serra E., Troestler C., *Constant sign and sign changing NLS ground states on noncompact metric graphs.* ArXiV preprint: https://arxiv.org/abs/2306.12121.

Overviews of the subject

- Adami R. *Ground states of the Nonlinear Schrodinger Equation on Graphs: an overview (Lisbon WADE).* https://www.youtube.com/watch?v=G-FcnRVvoos (2020)
- Riccardo Adami, Enrico Serra, and Paolo Tilli. Nonlinear dynamics on branched structures and networks. *Riv. Math. Univ. Parma (N.S.)*, 8(1):109–159, 2017.
- Kairzhan A., Noja D., Pelinovsky D. *Standing waves on quantum graphs.* J. Phys. A: Math. Theor. 55 243001 (2022)

An application: atomtronics

- A *boson*[1] is a particle with integer spin.

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- Since 2000: emergence of *atomtronics*, which studies circuits guiding the propagation of ultracold atoms.

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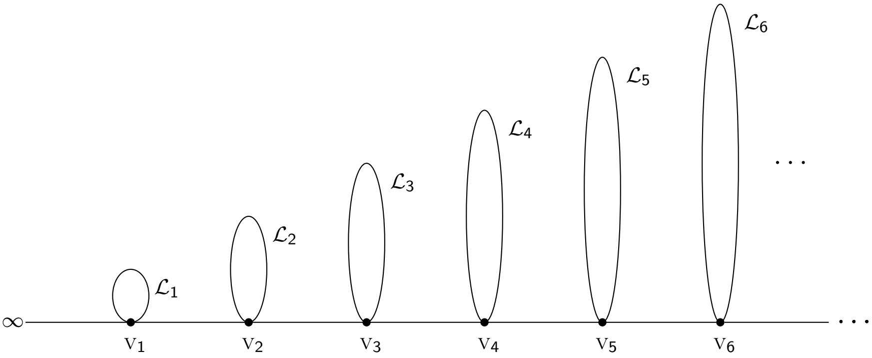
What's going on in case A2?

$\mathcal{J}_{\mathcal{G}}(\lambda) = \sigma_\lambda(\mathcal{G})$ and neither infima is attained



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- One obtains

$$s_\lambda = \mathcal{J}_{\mathcal{G}}(\lambda) \leq \sigma_\lambda(\mathcal{G}) \leq \liminf_{n\to\infty} c_\lambda(\mathcal{G}, \mathcal{L}_n) = s_\lambda,$$

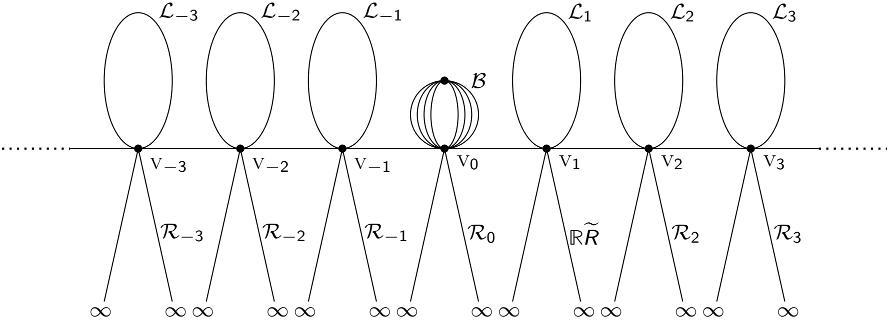
so

$$\mathcal{J}_{\mathcal{G}}(\lambda) = \sigma_\lambda(\mathcal{G}) = s_\lambda$$

and neither infimum is attained.

What's going on in case B2?

$\mathcal{J}_{\mathcal{G}}(\lambda) < \sigma_\lambda(\mathcal{G})$ and neither infima is attained

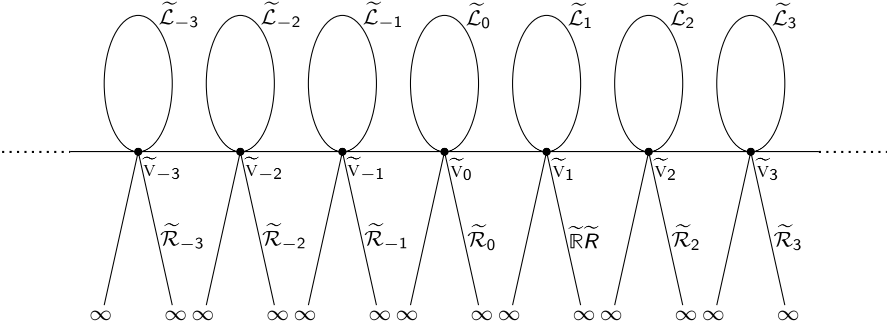


The graph $\mathcal{G}_N$.

The loops $\mathcal{L}_i$ have length N and $\mathcal{B}$ is made of N edges of length 1.

What's going on in case B2?

A second, periodic, graph



The graph $\widetilde{\mathcal{G}}_N$.

The loops $\widetilde{\mathcal{L}}_i$ have length N.

What's going on in case B2?

Two problems at infinity

- Since $\mathcal{G}_N$ and $\widetilde{\mathcal{G}}_N$ satisfy (H) and contain halflines, one has

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  but that $\sigma_\lambda(\mathcal{G}_N)$ *is not attained*.

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  but that $\sigma_\lambda(\mathcal{G}_N)$ *is not attained*.
- Therefore, for large N, we have that

$$s_\lambda = c_\lambda(\mathcal{G}_N) < \sigma_\lambda(\mathcal{G}_N),$$

  and neither infima is attained, as claimed.